Transformation Between Reference Ellipsoids, Using non-Euclidean Relationships

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Key words: geodesy, transformation, projections

SUMMARY

Transformation between different geodetic datums is a general task, e.g. if a Global Navigation Satellite Systems' receiver is used for the measurements. This paper does not deal with the transformation between datums, but the transformation between reference ellipsoids (as a part of the geodetic datum) is studied.

Reference ellipsoids (or spheres) are the basis of any projection system. A projection system describes the connection between the ellipsoidal, geographic coordinates (latitude and longitude) and the planar coordinates of the projection. If the relationship between the two reference surfaces is determined, transformation between two projection systems can be easily carried out based on this relationship (because the projections are known).

The developed method for determining the above mentioned relationship between two reference surfaces is based on projective geometry. Reference surfaces in geodesy (e.g. ellipsoids, spheres etc.) are quadrics. Handling of quadrics and their relationships in projective geometry provides a flexible solution for determining the relationship between two reference surfaces. Usage of collineations and correlations (as basic transformations in projective geometry) showed excellent and reliable results for transformation between two projection systems.

Theoretical background of the method, results of testing on different Hungarian geodetic datums, and further development plans are presented and studied in the paper.

SUMMARY (optional summary in one other language in addition to English, e.g. your own language)

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1. INTRODUCTION

In geodesy and surveying a paradigm change took place during the last decades. Field measurements were replaced by direct positioning, therefore the traditional observations, like angles and distances, were pushed back, by new techniques e.g. GNSS positioning on the field.
Projective geometry is the geometry of a ruler, counter to Euclidean geometry, in which constructions can be made with a ruler and a compass. In projective geometry there are no angles, no parallels, no distances, only intersections, coincidence etc. Traditionally projective relations are used in photogrammetry. It resembles nature, since the mathematic model of photogrammetry, and the constructions of photogrammetric instruments are based on projective geometry.
But projective geometry is an axiomatic based, independent geometry, one of the non-Euclidean geometries. It has a lot of opportunities in the usage in another part of our science. Since nowadays geodetic and surveying activities focus on direct positioning, potential use of projective geometry becomes more important.
The extent of this paper is limited, therefore some relations were not discussed deeply, only specified as definitions. All studied relationships were verified, if the reader needs more detailed deduction, please contact with the author.
If anyone is interested in projective geometry, Coxeter's books and papers are really a good starting point (Coxeter, 1973, Coxeter 1987)
Since projective geometry, and its relationships are not widely known the first section of this paper describes the definitions, required for the use of the developed transformation algorithm. In the second section the algorithm itself is discussed. The last section deals with the results of the developed method.

2. PROJECTIVE GEOMETRY

Projective geometry is dealing with the features, which are not changed by projection. Therefore, such things, like angles, distances, are uninteresting quantities, because they could be changed by a projection. But projective geometry can be taken into account as a "general geometry". Coxeter wrote: "*It is true, that projective geometry includes the affine, Euclidean and non-Euclidean geometries, but does not include the general Riemannian geometry, nor topology".* (Coxeter, 1987, p. 230).

2.1 Projective coordinate systems

In projective geometry homogenous coordinates are used. Homogenous coordinates are a type of barycentric coordinates.
In 2D space (on a plane) a point is determined by 3 homogenous coordinates:

$$P = \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \text{(2.1.1.)}$$

Since, these are barycentric coordinates, if:

$$R = \begin{bmatrix} \lambda X \\ \lambda Y \\ \lambda W \end{bmatrix}, \ P \equiv R, \quad \lambda \neq 0 \text{ (2.1.2)}$$

In 2D, the lines also have coordinates
Coordinates of the line, passes points P and R (P and R are different), as follows:

$$X_L = Y_P W_R - W_P Y_R$$
$$Y_L = W_P X_R - X_P W_R$$
$$W_L = X_P Y_R - Y_P X_R$$

(2.1.3)

From (2.1.3) it is clear, that if a point (P) lies on a line (L):

$$P^T L = 0.$$

(2.1.4)

P^T – is the transpose vector of P point,
L – is the vector of L line.

A homogenous projective coordinate system in 2D can be defined by 3 points (1,0,0), (0,1,0), (0,0,1) (which construct a triangle) and 1, called unit point (1,1,1), which is the centroid of these 3 points. By this construction we are able to describe any point (and line) in 2D, including the points at infinity as well. Based on these four points are required to define a homogenous projective coordinate system in 2D.
From this analytical construction one of the most beautiful characteristic of projective geometry is appeared, called duality. Duality means that in 2D we can exchange the words "point" and "line" in any theorem, the theorem remains true. It is a fantastic symmetry in projective geometry.
Definition of homogenous projective coordinate system in 3D is analogous to the 2D case. Four general points are needed, which construct a tetrahedron (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) respectively and 1 unit point (1,1,1,1), which is the centroid of the points of the tetrahedron.
In 3D space the points have 4 homogenous coordinates (X,Y,Z,W), and (analogously to 2D case) the planes have 4 homogenous coordinates). In 3D projective geometry the "point" and "plane" are dual constructions.

2.2 Collineation

2D collineation can be defined as:
Let there be a plane S_1, and 4 points on this plane, which construct a quadrangle A, B, C, D.
Let there be another plane S_2, and 4 point on this plane, which construct a quadrangle, A', B', C' and D'. There is one and only one collineation, which maps A, B, C, D to A', B', C' and D'. (Fig. 1.) (Szász, 1977).

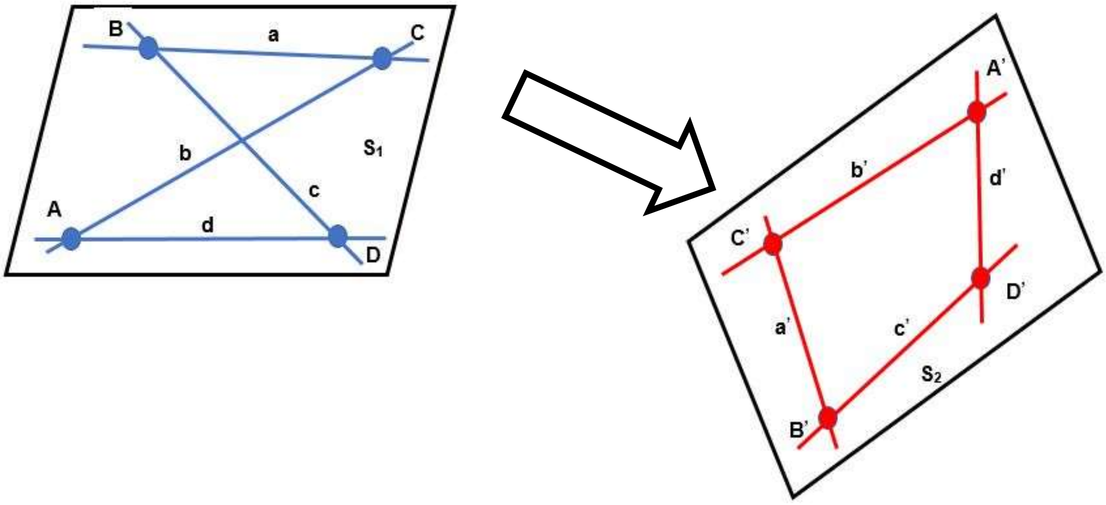

Fig. 1.: Collineation in 2D

In 3D, the collineation is analogue to 2D case, except five, general position points are needed. Collineation can be described by homogenous coordinates, as well:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ W \end{bmatrix} = \lambda \begin{bmatrix} X' \\ Y' \\ W' \end{bmatrix} \quad \lambda \neq 0, Det(C) \neq 0 \tag{2.2.1}$$

In 3D it is similar, but the dimension of collineation matrix is (4x4).
Determinant of collineation matrix should be non-zero, because in this case the collineation will be degenerative.

2.3. Correlation and Polarity

2D correlation can be defined as:
Let there be a plane S_1, and 4 points on this plane, which construct a quadrangle A, B, C, D.
Let there be another plane S_2, and 4 lines on this plane, which construct a quadrilateral, a', b', c' and d'. There is one and only one correlation, which maps A, B, C, D to a', b', c' and d'. (Fig. 2.) (Szász, 1977).

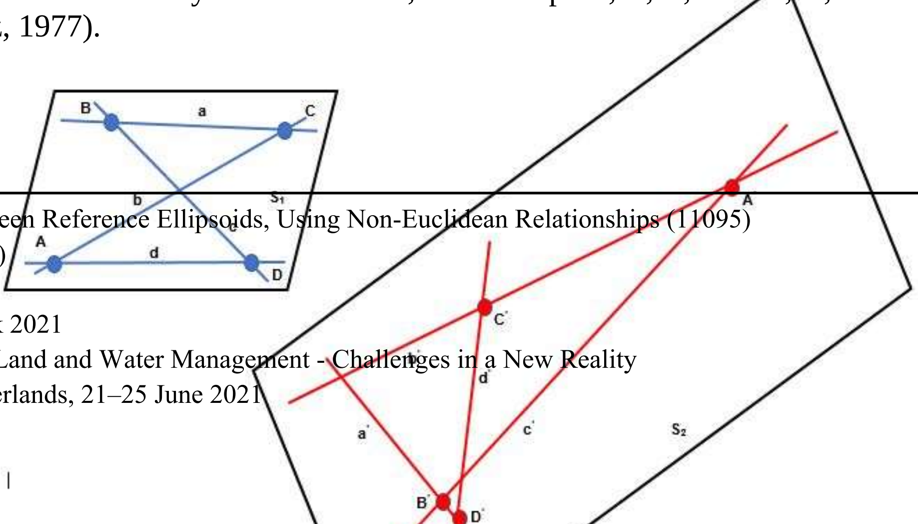

Fig.2.: Correlation in 2D

In 3D, the correlation is analogue to 2D case, but instead of four five general position points and planes are needed.

Collineation can be described by homogenous coordinates, as well:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ W \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \quad \lambda \neq 0, Det(C) \neq 0 \tag{2.3.1}$$

In 3D it is similar, but the dimension of correlation matrix is (4x4).

Determinant of correlation matrix should be non-zero, because in this case the correlation will be degenerative.

Polarity is a correlation of period 2. In general correlation transforms a point A into a line a', and transforms this line into a new point A''. When this correlation is a period of 2, A'' coincides with A. (Coxeter, 1987).

If a polarity transforms a point ”A” into a line ”a”, and ”A” lies on the ”a”, point ”A” is a self-conjugated point, and line ”a” is a self-conjugated line. Why these self conjugated points are important?

In a homogenous 2D coordinate system, if point ”A” is self-conjugated:

$$A^T C A = 0, \qquad Det(C) \neq 0, \tag{2.3.2}$$

where:

A – is the vector of self-conjugated point

C – is the matrix of polarity.

We should notice, that the matrix of a polarity in 2D is a symmetric matrix. Self-conjugated points (or lines) in a polarity define a conic.

2.4.Quadrics

In projective geometry there are no ellipsoids, hyperboloids, paraboloids, just only quadrics. Making differences among these quadrics is the mission of affine, or Euclidean geometry.
A quadric in 3D projective space can be defined as a set of self-conjugated points (planes) in a 3D polarity. The equation is similar to (2.3.2), but the dimension of matrix is (4x4), as the vectors of points(planes):

$$A^T CA = 0, \qquad Det(C) \neq 0, \tag{2.4.1}$$

But quadrics can be constructed in a different way, which is the key point of the developed method.

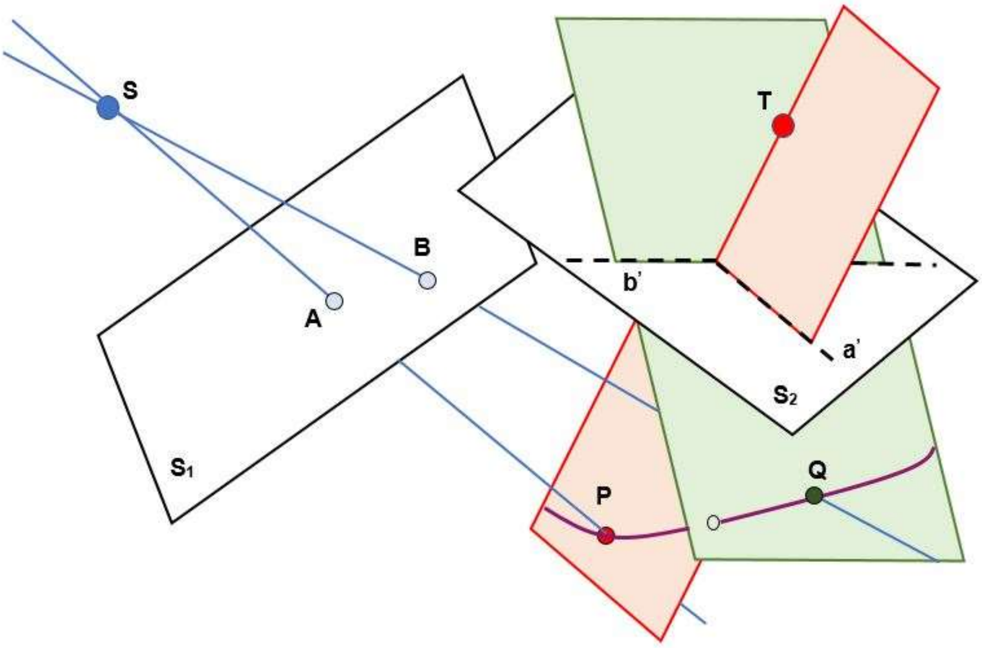

Fig.3.: Quadric as a construction of projective bundles

- Let there be a bundle of rays passes point S (Fig.3.). Let there be a bundle of planes passes point T.
- Let us intersect the bundle passes S by a plane S_1. The intersection of the bundle and the plane defines a point-field on plane S_1.
- Let us intersect the bundle passes T by a plane S_2. The intersection of the bundle and the plane defines a ray-field (a set of lines) on plane S_2.
- Let us establish a projective correlation between the point-field on plane S_1 and the ray-field on plane S_2.

The intersection points of the rays passes S, and of the corresponding planes passes T are on a quadric.
Using analytical tools:

Let there be S point's coordinates as (0,0,0,1) and T as (1,0,0,0).
Let there be S_1 plane's coordinates as [0,0,0,1] and S_2 plane's coordinates as [1,0,0,0].
The correlation between S_1 and S_2, as follows:

$$\begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \lambda \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \lambda \neq 0 \tag{2.4.2}$$

where:
X,Y,Z – are the coordinates of points in the point-field on S_1.
P,Q,R – are the coordinates of corresponding lines in the ray-field on S_2.
c_{ij} – are the elements of correlation matrix between the two planes.
Then the qaudric of this constuction can be defined by the following 3D polarity (Szász, 1977):

$$A = \begin{bmatrix} 0 & c_{21} & c_{31} & c_{41} \\ c_{21} & 2c_{22} & (c_{23}+c_{32}) & c_{42} \\ c_{31} & (c_{23}+c_{32}) & 2c_{33} & c_{43} \\ c_{41} & c_{42} & c_{43} & 0 \end{bmatrix} \tag{2.4.3}$$

3. TRANSFORMATION METHOD BETWEEN REFERENCE ELLIPSOIDS

3.1. Reconstruction of reference ellipsoids

Ellipsoids are quadrics. As it was described in chapter 2.4, quadrics can be represented as a set of self-conjugated points (planes), like (2.4.1).
A reference ellipsoid can be described by lenght of its semi-major and semi-minor axises.
Let us assume, that the 3D coordinate system of the ellipsoid (X,Y,Z) is defined on the traditional way.
Let us define a homogenous, 3D projective coordinate system, in which:

- (0,0,0,1) point concides with the centre of the ellipsoid,
- (1,0,0,0) point coincides with the point at infinity on X-axis,
- (0,1,0,0) point coincides with the point at infinity on Y-axis,
- (0,0,1,0) point coincides with the point at infinity on Z-axis, and
- (1,1,1,1) point coincides with the point (1,1,1) in the coordinate system.

In this construction any point in the original (Euclidean) coordinate system (X,Y,Z) can be describe as (X,Y,Z,1).
In such a construction the 3D polarity, which defines the quadric (the ellipsoid) is the follows:

$$P = \begin{bmatrix} b^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & -a^2b^2 \end{bmatrix} \tag{3.1.1}$$

where a, is the lenght of semi-major axis, and b is the lenght of semi-minor of the ellipsoid.
Any point, which is on the surface of the quadric, as self-conjugated point, satisfies the equation (2.4.1).

Let us choose 2, different points on the surface of the quadric, and transform them by a 3D collineation into a coordinate system, in which their coordinates are (0,0,0,1) and (1,0,0,0) respectively (2.4) . In order to make this solution simplier, these 2, choosen points are (a,0,0,1) and (0,0,b,1). From the set of 3D collineations, which satisfy these requirements, the following were choosen:

$$L = \begin{bmatrix} -1 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ b & 0 & a & -ab \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.1.2}$$

Then the original polarity (3.1.1) was transformed by (3.1.2) collineation into the new base, the result is the following:

$$R = \begin{bmatrix} 0 & 0 & a^2b & -a^2b^2 \\ 0 & b^2 & 0 & 0 \\ a^2b & 0 & a^2 & 0 \\ -a^2b^2 & 0 & 0 & 0 \end{bmatrix} \tag{3.1.3}$$

But this R polarity should be the same, as it was defined in (2.4.3).
By this way, the reconstruction of the ellipsoid in the new base is completed.
The parameters of the correlation between S_1 and S_2 (see 2.4), are:

$$C = \begin{bmatrix} 0 & \frac{b^2}{2} & 0 \\ a^2b & 0 & \frac{a^2}{2} \\ -a^2b^2 & 0 & 0 \end{bmatrix}. \tag{3.1.4}$$

3.2. Transformation steps

Tranformation from Ellipsoid E to ellipsoid F.

Initial data:

- Semi major and semi-minor axeses of ellipsoids (Ellipsoid E, and Ellipsoid F).
- 3D coordinated of homologous points on the ellipsoids (P_E and P_F), int the traditional, 3D Euclidean coordinate system.

Steps:

1. Reconstruction of ellipsoid E and ellipsoid F by the method, described in (3.1). Output: C_E and C_F correlation matrices as (3.1.4).
2. Transformation of homologous points on E and F into the new base, by L_E and L_F (3.1.2). Output: Homogenous coordinates of homologous points in the new base.
3. Calculation of coordinates of point fields on plane S_1 (see 2.4). Output: point field on S_{1E} and S_{1F}.
4. Determine the collineation between S_{1E} and S_{1F}. Output: C_{EF} the collineation matrix between S_{1E} and S_{1F}.

Calculation of transformed coordinates:

1. Calculate any point's coordinates in ellipsoid E point-field by step 2 and 3.
2. Transform this point by C_{EF} collineation into the ellipsoid F point-field.
3. Determine the coordinates of the transformed point by C_F correlation matrix.

4. Transform this point back to the original, Euclidean coordinate system by the inverse of L_F.
5. Transformation is completed.

4. Results

For testing the above mentioned algorithms the following networks have been used:
FAGH – Atrogeodetic Network of the Former Socialist Countries, covering Hungary. Reference ellipsoid: S42 (Krasowsky ellipsoid). Geodetic Datum: S42/52 (166 points)
EAGH83 - Atrogeodetic Network of the Former Socialist Countries, covering Hungary. Reference ellipsoid: S42 (Krasowsky ellipsoid). Geodetic Datum: S42/83 (166 points)
HD72 – Official Highest Order Network of Hungary. Reference ellipsoid: IUGG GRS 1967. Geodetic Datum: Hungarian Datum 1972. (166 points)
ED87 – European Datum 1987 covering Hungary. Reference ellipsoid: Hayford 1924. (148 points)
ETRF2000 – European Terrestrial Reference Frame 2000 covering Hungary. Reference ellipsoid: GRS 1980. (66 points)

Adjustment

Because of the chracteristics of transformation, the goal of adjustment has been the determination a collineation parameters (C_{EF}, see 3.2) between the two point-fields.
For adjustmentI the well-known least squares method were used, by an own-developed software solution.
For every combination of ellipsoids 3 adjustments have been carried out, for 13, 56 and 134 homologous points (of course for ETRF2000 points 134 points adjustment has not been used, because it has only 66 points).
After the determination of collineation parameters, the transformed point coordinates have been calculated by the method, described in 3.2.
Maximum and avarage discrepancies after adjustment were calculated by the following formula:

$$d = \sqrt{d_X^2 + d_Y^2 + d_Z^2}$$

where:

$$d_X = X_F - X_E^{TR}$$
$$d_Y = Y_F - Y_E^{TR}$$
$$d_Z = Z_F - Z_E^{TR}$$

(X_F, Y_F, Z_F) – are the coordinates of the original point on ellipsoid F
$(X_E^{TR}, Y_E^{TR}, Z_E^{TR})$ – are the coordinates of the transformed point from ellipsoid E.

On Figure 4. an adjusted collineation matrix is presented, which describes the collineation from ED87 to ETRF2000.

```
Kiegyenlitett Kollineacios matrix:

1.00000000e+00 -1.87981494e-12 -2.69411406e-12
-5.85055933e+01 9.99980086e-01 -2.55255249e-05
-6.95921423e-01 -3.86467956e-06 9.99982640e-01
```

Fig. 4: Adjusted collieation matrix from ED87 to ETRF2000

The number of parameters in adjustment is 8, because of the mathematical model, so 1 parameter can be freely fixed. In this case collineation parameter c_{11} was fixed as 1. The reason is really simple, can be derived from the characteristic of a collineation, and the use of homogenous coordinates:

$$C \equiv \lambda C \quad \lambda \neq 0$$

Gross error detection has not been used during the adjustment. It may help to reach even more accurate transformationshould be needed in further investigation.
Another question also rises: Is there any metric meaning of collineation parameters(Figure 4.)? Currently nothing is suggesting so, but further investigation is required to answer this question correctly.

The results of transformations summarized in Table 1.
In second column the number of points included in adjustment can be found.
In the third and fourth column the maximum and avarage discrepancies are shown for the points, which have not been included in adjustment.
In fifth and sixth column the maximum and avarage discrepancies are shown for the points, which have been included in adjustment.

From/To	No. points in Adj.	Discr. [cm] Maximum	Discr. [cm] Average	Discr. [cm] Adj. Maximu	Discr. [cm] Adj. Average
ETRF2000/HD72	**13**	**30.3**	**11.5**	**27.9**	**11.2**
ETRF2000/HD72	**56**	**31.2**	**10.6**	**31.2**	**10.2**
FAGH/ETRF2000	**13**	**36.2**	**14.6**	**36.2**	**13.3**
FAGH/ETRF2000	**56**	**40.4**	**13.5**	**40.4**	**12.9**
FAGH/HD72	**13**	**21.8**	**5.0**	**8.6**	**3.8**
FAGH/HD72	**56**	**22.6**	**4.9**	**10.2**	**3.6**
FAGH/HD72	**134**	**18.3**	**4.2**	**12.6**	**3.5**
FAGH/ED87	**13**	**32.7**	**7.6**	**20.0**	**7.1**
FAGH/ED87	**56**	**31.6**	**7.5**	**19.0**	**6.3**
FAGH/ED87	**134**	**30.6**	**7.3**	**30.6**	**7.1**
FAGH/EAGH83	**13**	**51.7**	**12.9**	**19.7**	**9.5**
FAGH/EAGH83	**56**	**48.3**	**11.6**	**22.0**	**8.1**
FAGH/EAGH83	**134**	**48.8**	**11.2**	**33.6**	**9.4**
EAGH83/HD72	**13**	**36.1**	**13.4**	**14.7**	**10.1**
EAGH83/HD72	**56**	**35.9**	**12.1**	**22.4**	**9.2**
EAGH83/HD72	**134**	**32.8**	**11.3**	**31.2**	**9.6**
EAGH83/ED87	**13**	**33.4**	**10.1**	**18.7**	**9.8**
EAGH83/ED87	**56**	**36.2**	**9.2**	**26.5**	**8.7**
EAGH83/ED87	**134**	**35.5**	**9.1**	**35.5**	**8.8**
EAGH83/ETRF2000	**13**	**34.1**	**16.5**	**20.4**	**13.3**
EAGH83/ETRF2000	**56**	**32.8**	**14.6**	**32.3**	**13.8**
ED87/HD72	**13**	**30.1**	**8.7**	**19.6**	**7.7**
ED87/HD72	**56**	**31.9**	**8.6**	**18.7**	**7.3**
ED87/HD72	**134**	**28.6**	**8.1**	**28.6**	**8.0**
ED87/ETRF2000	**13**	**31.8**	**12.3**	**24.7**	**9.5**
ED87/ETRF2000	**56**	**38.0**	**11.4**	**38.0**	**11.0**

Table 1.: Spatial discrepancies after transformation

Investigation of the results, shown in Table 1., has shown that this developed transformation method provides geodetic accuracy for the transformation of reference ellipsoids for a whole country by only 8 parameters. (Territory of Hungary is 93 000 km^2).
Some cases adjustment did not decrease the maximum discrepancies, but for the average discrepancies did it (it derives from the charactarestics of least squares method).
Using gross error detection and other statistical tools, or other estimation method (e.g. robust estimators) may provide better results, this could be a point of further development.

5. CONCLUSIONS

In this paper we described a non-Euclidean solution for transformation between reference ellipsoids. Because of the not widely known projective relationships, the basic projective definitions were discussed.
The developed method, based on projective geometry, showed geodetic accurate results for transformation. Because of such a new approximation of transformation, further testing and investigation is needed, but the first tests showed excellent results.
Direct positioning mostly replaced the traditional angle and distance measurement in geodesy and surveying, position is the most important nowadays. This fact could cause result in the rise of such a fantastic building of human thought: projective geometry, and other non-Euclidean geometries in geodesy and surveying.
At the end let us to cite the sentences of Harold Scott Macdonald Coxeter:
"*Möbius's invention of homogenous coordinates was one of the most farreaching ideas in the history of mathematics, comparable to Leibniz's invention of differentials, which enabled him to express the equation*

$$\frac{d}{dx}f(x) = f'(x).$$

in the homogenous form

$$df(x) = f'(x)dx$$

(for instance, $d \sin x = \cos x \, dx$*)."* (Coxeter, 1987, p. 221)

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BIOGRAPHICAL NOTES

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